

Affine Transformations of Loop Nests for Parallel Execution and Distribution of Data over Processors ^{*}

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Abstract. The paper is devoted to the problem of mapping affine loop nests onto distributed memory parallel computers. A method to find affine transformations of loop nests for parallel execution and distribution of data over processors is presented. The method tends to minimize the number of communications between processors and to improve locality of data within one processor. A problem of determination of data exchange sequence is investigated. Conditions to determine the ability to arrange broadcast is presented.

1 Introduction

To map algorithms given by sequential programs onto distributed memory parallel computers is to distribute data and computations to processors, to determine an execution sequence of operations and a data exchange sequence. The most important problems are: scheduling [1], space-time mapping [2], alignment [1,3,4], determination of data exchange sequence [5]. An essential stage of the solution of these problems is to find functions (scheduling functions, statement and array allocation functions) satisfying certain constraints.

One of the preferable parallelization schemes is based on obtaining multi-dimensional scheduling functions. Some coordinates of the multi-dimensional scheduling functions are used for operations allocation. The other coordinates are used for scheduling operations.

For the program execution time to be as small as possible it is necessary to solve an alignment problem. It consists in coordinated operations and data allocation to minimize the communications.

The program execution time depends not only on the execution time of operations but also on the memory access time. The access time depends on data location in the hierarchical memory. Therefore the problem of prompt data reuse within one processor (localization problem) is of great importance [6]. Two kinds

^{*} Supported by the Belorussian Foundation for Basic Research under Grant F05-019

of localization, such as localization in time and in space, are available. The time localization is used to set the operations execution sequence so that the data be reused before it is moved to a lower level memory. The spatial localization allows to use the data allocated close to each other in memory. The locality depends on the execution sequence of operations. Hence, it is desirable to take it into account when scheduling functions are obtained.

In this paper, a method of simultaneous solution of all mentioned problems is suggested. It results in high performance of parallel code execution.

After transformation of an algorithm for parallel execution it is necessary to determine the data exchange sequence. In many cases utilization of broadcast, gather, scatter, reduction, and translation enables to improve the efficiency of a parallel program. In this paper, we investigated the problem of determination of the data exchange sequence. We suggest the conditions to determine the case when the broadcast communications may be used.

2 Main Definitions

Let an algorithm be represented by an affine loop nest. For such algorithms, array indices and bounds of loops are affine functions of outer loop indices or loop-invariant variables. Let a loop nest contain K statements S_β and use L arrays a_l . By V_β denote the iteration domain of statement S_β , by W_l denote the index domain of array a_l . By n_β denote a number of loops surrounding statement S_β . By ν_l denote dimension of array a_l . Then $V_\beta \subset \mathbb{Z}^{n_\beta}$, $W_l \subset \mathbb{Z}^{\nu_l}$. By $J \in \mathbb{Z}^{n_\beta}$ denote the iteration vector, by $N \in \mathbb{Z}^e$ denote the vector of outer variables, e is the number of these variables.

Let $\bar{F}_{l,\beta,q}: V_\beta \rightarrow W_l$ denote access function that puts the iteration domain V_β into correspondence with the index domain W_l for the q -th input of elements of array a_l into instruction S_β . Suppose $\bar{F}_{l,\beta,q}$ are affine functions: $\bar{F}_{l,\beta,q}(J) = F_{l,\beta,q}J + G_{l,\beta,q}N + f^{(l,\beta,q)}$, where $J \in V_\beta$, $F_{l,\beta,q} \in \mathbb{Z}^{\nu_l \times n_\beta}$, $N \in \mathbb{Z}^e$, $G_{l,\beta,q} \in \mathbb{Z}^{\nu_l \times e}$, $f^{(l,\beta,q)} \in \mathbb{Z}^{\nu_l}$.

Given a statement S_β , a computation instance of S_β is called an operation and is denoted by $S_\beta(J)$. Denote a dependence of operation $S_\beta(J)$ from operation $S_\alpha(I)$ by $S_\alpha(I) \rightarrow S_\beta(J)$. We consider flow-, anti-, out-, and in-dependences. Denote by P a set of pairs of indices such that $S_\alpha(I) \rightarrow S_\beta(J)$.

Let $\bar{\Phi}_{\alpha,\beta}: V_{\alpha,\beta} \rightarrow V_\alpha$ be dependence function. If $S_\alpha(I) \rightarrow S_\beta(J)$, $I \in V_\alpha$, $J \in V_{\alpha,\beta} \subseteq V_\beta$, then $I = \bar{\Phi}_{\alpha,\beta}(J)$. Suppose $\bar{\Phi}_{\alpha,\beta}$ are affine functions: $\bar{\Phi}_{\alpha,\beta}(J) = \Phi_{\alpha,\beta}J + \Psi_{\alpha,\beta}N - \varphi^{(\alpha,\beta)}$, $J \in V_{\alpha,\beta}$, $\Phi_{\alpha,\beta} \in \mathbb{Z}^{n_\alpha \times n_\beta}$, $\varphi^{(\alpha,\beta)} \in \mathbb{Z}^{n_\alpha}$, $N \in \mathbb{Z}^e$, $\Psi_{\alpha,\beta} \in \mathbb{Z}^{n_\alpha \times e}$.

3 Multi-Dimensional Scheduling Functions. Data Allocation Functions

Let $n = \max_{1 \leq \beta \leq K} n_\beta$. Let functions $\bar{t}^{(\beta)}: V_\beta \rightarrow \mathbb{Z}^n$ assign a vector $(t_1^{(\beta)}(J), \dots, t_n^{(\beta)}(J))$ to each operation $S_\beta(J)$. Suppose $t_\xi^{(\beta)}$ are affine functions: $t_\xi^{(\beta)}(J) =$

$\tau^{(\beta,\xi)}J + b^{(\beta,\xi)}N + a_{\beta,\xi}, \quad J \in V_\beta, \tau^{(\beta,\xi)} \in \mathbb{Z}^{n_\beta}, b^{(\beta,\xi)}, N \in \mathbb{Z}^e, a_{\beta,\xi} \in \mathbb{Z}, 1 \leq \beta \leq K, 1 \leq \xi \leq n.$

Functions $\bar{t}^{(\beta)}$ are called vector scheduling functions if

$$\text{rang } T^{(\beta)} = n_\beta, \quad 1 \leq \beta \leq K, \quad (1)$$

$$\bar{t}^{(\beta)}(J) \geq_{lex} \bar{t}^{(\alpha)}(I), \quad J \in V_\beta, I \in V_\alpha, \text{ if } S_\alpha(I) \rightarrow S_\beta(J). \quad (2)$$

Here $T^{(\beta)}$ is a matrix whose rows are vectors $\tau^{(\beta,1)}, \dots, \tau^{(\beta,n)}, S_\alpha(I) \rightarrow S_\beta(J)$ is any dependence except in-dependence, notation \geq_{lex} denotes "lexicographically greater or equal to".

A set of vector functions $\bar{t}^{(\beta)}, 1 \leq \beta \leq K$, is called a multi-dimensional scheduling. We can use these functions to transform loops assuming the operation $S_\beta(J)$ to be executed at the iteration $\bar{t}^{(\beta)}(J)$. Thus we interpret elements of the vector $\bar{t}^{(\beta)}$ as indices of the transformed loop nest for the statement S_β : $t_1^{(\beta)}$ is the index of the outermost loop, $t_n^{(\beta)}$ is the index of the innermost loop. Note that the functions $\bar{t}^{(\beta)}$ determine permissible transformation of the loop nest, i.e., this transformation keeps the execution sequence of dependent operations.

We consider functions $(t_1^{(\beta)}, \dots, t_r^{(\beta)})$, $r < n$, as allocation functions that determine spatial mapping of an algorithm to r -dimensional space of virtual processors. That is, the values of indices of r external loops of the transformed algorithm determine processor coordinates. The values of indices of $n-r$ internal loops determine iterations to be executed on the processor.

Usually we need to take into account the number of processors used to execute the program. Then to simplify code generation it is necessary that the following conditions be valid $t_\xi^{(\beta)}(J) \geq t_\xi^{(\alpha)}(I)$, $J \in V_\beta, I \in V_\alpha$, if $S_\alpha(I) \rightarrow S_\beta(J)$, $1 \leq \xi \leq r$.

Let functions $\bar{d}^{(l)} : W_l \rightarrow \mathbb{Z}^r$ assign a vector $(d_1^{(l)}(F), \dots, d_r^{(l)}(F))$ to each element $a_l(F)$ of an array a_l . Suppose $d_\xi^{(l)}$ are affine functions: $d_\xi^{(l)}(F) = \eta^{(l,\xi)}F + z^{(l,\xi)}N + y_{l,\xi}$, $F \in W_l, \eta^{(l,\xi)} \in \mathbb{Z}^{n_l}, z^{(l,\xi)}, N \in \mathbb{Z}^e, y_{l,\xi} \in \mathbb{Z}, 1 \leq l \leq L, 1 \leq \xi \leq r$. Let element $a_l(F)$ be stored in the local memory of the processor determined by the coordinates $(d_1^{(l)}(F), \dots, d_r^{(l)}(F))$.

Let us introduce some notation: $\tilde{\tau}^{(\xi)} = (\tau^{(1,\xi)}, \dots, \tau^{(K,\xi)}, \eta^{(1,\xi)}, \dots, \eta^{(L,\xi)}, b^{(1,\xi)}, \dots, b^{(K,\xi)}, z^{(1,\xi)}, \dots, z^{(L,\xi)}, a_{1,\xi}, \dots, a_{K,\xi}, y_{1,\xi}, \dots, y_{L,\xi})$ is a vector, whose entries are parameters of functions $t_\xi^{(\beta)}$ and $d_\xi^{(l)}$.

The following proposition gives the condition to be used for finding scheduling functions that satisfy condition (1).

Proposition 1. Suppose $\text{rang } T_{1:\xi-1}^{(\beta)} = r$, $r < n_\beta$, where $T_{1:\xi}^{(\beta)}$ is a matrix whose rows are vectors $\tau^{(\beta,i)}, 1 \leq i \leq \xi$. Suppose $s_\beta^{(\xi)}$ is a fixed vector of a set $S_\beta^{(\xi)} = \{s \in \mathbb{Z}^{n_\beta} \mid \tau^{(\beta,i)}s = 0, 1 \leq i \leq \xi-1, s \neq 0\}$, $2 \leq \xi \leq n$. Then $\text{rang } T_{1:\xi}^{(\beta)} = r+1$ if $\tau^{(\beta,\xi)}s_\beta^{(\xi)} \neq 0$.

Condition $\tau^{(\beta,\xi)} s_\beta^{(\xi)} \neq 0$ is equivalent to the following inequality in the vector form

$$|\tilde{\tau}^{(\xi)} \tilde{s}_\beta^{(\xi)}| \geq 1. \quad (3)$$

Let $v^{(\alpha,\beta,m)}$ be vertices of the polyhedron $V_{\alpha,\beta}$, $m(\alpha,\beta)$ be the number of the vertices. Any vertex $v^{(\alpha,\beta,m)}$ can be represented in the form $v^{(\alpha,\beta,m)} = R^{(\alpha,\beta,m)}N + \omega^{(\alpha,\beta,m)}$. Let $N^{(0)} \in \mathbb{Z}^e$ be a vector whose i -th entry is equal to the smallest possible value of the outer variable N_i . Suppose coordinates of the vector N can be unlimited large. Then we can show that $t_\xi^{(\beta)}(J) - t_\xi^{(\alpha)}(I)$ is non-negative for all I and J such that $S_\alpha(I) \rightarrow S_\beta(J)$ iff

$$\begin{aligned} & ((\tau^{(\beta,\xi)} - \tau^{(\alpha,\xi)} \Phi_{\alpha,\beta}) R^{(\alpha,\beta,m)} + b^{(\beta,\xi)} - b^{(\alpha,\xi)} - \tau^{(\alpha,\xi)} \Psi_{\alpha,\beta}) N^{(0)} + \\ & + (\tau^{(\beta,\xi)} - \tau^{(\alpha,\xi)} \Phi_{\alpha,\beta}) \omega^{(\alpha,\beta,m)} + a_{\beta,\xi} - a_{\alpha,\xi} + \tau^{(\alpha,\xi)} \varphi^{(\alpha,\beta)} \geq 0, \quad 1 \leq m \leq m(\alpha,\beta); \\ & (\tau^{(\beta,\xi)} - \tau^{(\alpha,\xi)} \Phi_{\alpha,\beta}) R^{(\alpha,\beta,m)} + b^{(\beta,\xi)} - b^{(\alpha,\xi)} - \tau^{(\alpha,\xi)} \Psi_{\alpha,\beta} \geq 0, \quad 1 \leq m \leq m(\alpha,\beta). \end{aligned}$$

In the vector-matrix form

$$\tilde{\tau}^{(\xi)} D_{\alpha,\beta}^\varphi \geq 0, \quad \tilde{\tau}^{(\xi)} D_{\alpha,\beta} \geq 0. \quad (4)$$

Let introduce in the consideration vector variables $z_{\alpha,\beta}^\varphi$ and $z_{\alpha,\beta}$. The solution of (4) is the solution of equations

$$\tilde{\tau}^{(\xi)} D_{\alpha,\beta}^\varphi - z_{\alpha,\beta}^\varphi = 0, \quad z_{\alpha,\beta}^\varphi \geq 0, \quad \tilde{\tau}^{(\xi)} D_{\alpha,\beta} - z_{\alpha,\beta} = 0, \quad z_{\alpha,\beta} \geq 0. \quad (5)$$

The following propositions can be easily proved:

- 1) If $z_{\alpha,\beta}^\varphi = 0$, $z_{\alpha,\beta} = 0$ in (5) for all ξ , $1 \leq \xi \leq n$, then $\bar{t}^{(\beta)}(J) = \bar{t}^{(\alpha)}(I)$ for all I and J such that $S_\alpha(I) \rightarrow S_\beta(J)$.
- 2) If $z_{\alpha,\beta}^\varphi > 0$ in (5), then $t_\xi^{(\beta)}(J) - t_\xi^{(\alpha)}(I) > 0$ for all I and J such that $S_\alpha(I) \rightarrow S_\beta(J)$.

Thus, to find space-time mapping of an algorithm is to find vectors $\tilde{\tau}^{(\xi)}$, $1 \leq \xi \leq n$, which the following conditions are valid for. Suppose we are searching vectors $\tilde{\tau}^{(\xi)}$, $\xi = 1, 2, \dots, n$, sequentially. Then condition (3) has to be valid for β such that $n - \xi + 1 = n_\beta - \text{rang } T_{1;\xi-1}^{(\beta)}$. Conditions (5) have to be valid for all $(\alpha, \beta) \in P$ except the following case. Suppose $\xi \geq r + 1$ and for some $(\alpha, \beta) \in P$ the inequality $z_{\alpha,\beta}^\varphi > 0$ is valid, then validity of conditions (5) is not necessary for these (α, β) in the sequel.

Consider the alignment problem. The operation $S_\beta(J)$ is assigned to execute at the virtual processor $(t_1^{(\beta)}(J), \dots, t_r^{(\beta)}(J))$. The array element $a_l(\bar{F}_{l,\beta,q}(J))$ is stored in the local memory of the processor $(d_1^{(l)}(\bar{F}_{l,\beta,q}(J)), \dots, d_r^{(l)}(\bar{F}_{l,\beta,q}(J)))$. The expressions $\delta_\xi^{l,\beta,q}(J) = t_\xi^{(\beta)}(J) - d_\xi^{(l)}(\bar{F}_{l,\beta,q}(J))$, $1 \leq \xi \leq r$, determine the distance between the processors. Assuming $\delta_\xi^{l,\beta,q}(J) = 0$ we obtain conditions for communication-free allocation: $\tau^{(\beta,\xi)} - \eta^{(l,\xi)} F_{l,\beta,q} = 0$, $b^{(\beta,\xi)} - \eta^{(l,\xi)} G_{l,\beta,q} - z^{(l,\xi)} = 0$, $a_{\beta,\xi} - \eta^{(l,\xi)} f^{(l,\beta,q)} - y_{l,\xi} = 0$. In the vector-matrix form $\tilde{\tau}^{(\xi)} \Delta_{l,\beta,q}^F = 0$, $\tilde{\tau}^{(\xi)} \Delta_{l,\beta,q}^G = 0$, $\tilde{\tau}^{(\xi)} \Delta_{l,\beta,q}^f = 0$.

Introduce in the consideration vector variables $z_{l,\beta,q}^F$, $z_{l,\beta,q}^G$, $z_{l,\beta,q}^f$. Thus, to find operation and data allocation such that a number of communications is as

small as possible is to minimize (or to put to zero if it is possible) coordinates of the vectors $z_{l,\beta,q}^F, z_{l,\beta,q}^G, z_{l,\beta,q}^f$ which the following equations are valid for

$$|\tilde{\tau}^{(\xi)} \Delta_{l,\beta,q}^F| - z_{l,\beta,q}^F = 0, \quad |\tilde{\tau}^{(\xi)} \Delta_{l,\beta,q}^G| - z_{l,\beta,q}^G = 0, \quad |\tilde{\tau}^{(\xi)} \Delta_{l,\beta,q}^f| - z_{l,\beta,q}^f = 0 \quad . \quad (6)$$

Here $|v|$ is a vector whose entries are modules of entries of a vector v .

4 Conditions of Time and Space Localization

To obtain time localization is to find functions $\bar{t}^{(\beta)}$ so that values $\bar{t}^{(\beta)}(J)$ and $\bar{t}^{(\beta)}(I)$ satisfying (2) are as more lexicographically close to each other as it is possible (reuse of an array element is as more quicker as these values are closer). We reduced conditions (2) to constraints (5); thus, to achieve our goal is to minimize (to zero at best) vectors $z_{\alpha,\beta}^\varphi$ and $z_{\alpha,\beta}$.

Validity of condition (2), i.e., conditions (5) is necessary for all dependences except in-dependences. Write analogues of conditions (5) for in-dependences:

$$|\tilde{\tau}^{(\xi)} D_{\alpha,\beta}^\varphi| - z_{\alpha,\beta}^\varphi = 0, \quad |\tilde{\tau}^{(\xi)} D_{\alpha,\beta}| - z_{\alpha,\beta} = 0 \quad , \quad (7)$$

Thus, requirements of time localization can be reduced to vectors $z_{\alpha,\beta}^\varphi$ and $z_{\alpha,\beta}$ minimizing (zeroing if it is possible) when conditions (5), (7) are valid.

To obtain space localization is to use array elements that are stored close to each other in memory at the iterations that are close to each other. To be definite, assume that we use a programming language C. In this case, storing array elements is realized by rows. Thus, the l -th array elements that are stored close to each other in memory are those that differ from each other in the last coordinate of the index expressions: $\bar{F}_{l,\beta,q}(J) - \bar{F}_{l,\beta,q}(I) = \lambda e_{\nu_l}^{(\nu_l)}$, $\lambda \in \mathbb{Z}$. We realize space localization among operations of the same statement for the fixed access to array.

Introduce some notation: $\tilde{F}_{l,\beta,q} \in \mathbb{Z}^{(\nu_l-1) \times n_\beta}$ is a matrix whose rows are rows of the matrix $F_{l,\beta,q}$ except the last row; $r(l, \beta, q)$ is rang of the matrix $\tilde{F}_{l,\beta,q}$, $r(l, \beta, q) < n_\beta$; $d_{l,\beta,q}^{(\gamma)} = 0$, $1 \leq \gamma \leq n_\beta - r(l, \beta, q)$, is a fundamental system of solutions of a uniform system of equations $\tilde{F}_{l,\beta,q} x = 0$.

Theorem 1. *Let $\bar{t}^{(\beta)}$ be a multi-dimensional scheduling. Choose functions $t_\xi^{(\beta)}$, $\xi \in \{\xi_1, \dots, \xi_{r(l,\beta,q)}\}$, among functions $t_1^{(\beta)}, \dots, t_d^{(\beta)}$, $d < n$. Suppose these functions satisfy conditions*

$$\tau^{(\beta,\xi)} d_{l,\beta,q}^{(\gamma)} = 0 \quad , \quad 1 \leq \gamma \leq n_\beta - r(l, \beta, q) \quad , \quad (8)$$

$$\text{rang } T_{l,\beta,q} = r(l, \beta, q) \quad .$$

Here $T_{l,\beta,q}$ is a matrix whose rows are vectors $\tau^{(\beta,\xi_1)}, \dots, \tau^{(\beta,\xi_{r(l,\beta,q)})}$. Then elements of only one row of the l -th array are used in the q -th access of the operation S_β for fixed values of indices of d outer loops.

That is, to obtain space localization is to get $r(l, \beta, q)$ linear independent vectors $\tau^{(\beta, \xi)}$ that satisfy condition (8); values ξ are intended to be as small as possible. Condition (8) can be written in the vector-matrix form $\tilde{\tau}^{(\xi)} D_{l, \beta, q} = 0$.

Thus conditions of space localization can be reduced to vectors $z_{l, \beta, q}$ minimization (zeroing if it is possible) when the following conditions are valid

$$|\tilde{\tau}^{(\xi)} D_{l, \beta, q}| - z_{l, \beta, q} = 0 \quad . \quad (9)$$

5 Procedure of Affine Transformation of Loop Nests

Introduce some notation: $D_{\varphi}^{(1)}$ and $D^{(1)}$ are sets of matrices $D_{\alpha, \beta}^{\varphi}$ and $D_{\alpha, \beta}$ accordingly that describe flow-, out- and anti-dependences; $D_{\varphi, in}^{(1)}$ and $D_{in}^{(1)}$ are sets of matrices $D_{\alpha, \beta}^{\varphi}$ and $D_{\alpha, \beta}$ accordingly that describe in-dependences; D_F , D_G , D_f are sets of matrices $\Delta_{l, \beta, q}^F$, $\Delta_{l, \beta, q}^G$, and vectors $\Delta_{l, \beta, q}^f$ accordingly; $D_s^{(1)}$ is a set of matrices $D_{l, \beta, q}$; $S_{\beta}^{(1)} = \mathbb{Z}^{n_{\beta}}$; $T_{1:0}^{(\beta)} = 0^{(n_{\beta})}$; $T_{l, \beta, q}^{(\xi)}$ is a matrix whose rows are vectors $\tau^{(\beta, d)}$, $1 \leq d \leq \xi$, satisfying condition (8); $L^{(\xi)} = \{ \beta \mid n - \xi + 1 = n_{\beta} - \text{rang } T_{1:\xi-1}^{(\beta)} \}$; $\rho(z_{\alpha, \beta}^{\varphi}, z_{\alpha, \beta}, z_{l, \beta, q}^F, z_{l, \beta, q}^G, z_{l, \beta, q}^f, z_{l, \beta, q}) = \sum_{\alpha, \beta} (\lambda_{\alpha, \beta}^{\varphi} z_{\alpha, \beta}^{\varphi} + \lambda_{\alpha, \beta} z_{\alpha, \beta}) + \sum_{l, \beta, q} (\lambda_{l, \beta, q}^F z_{l, \beta, q}^F + \lambda_{l, \beta, q}^G z_{l, \beta, q}^G + \lambda_{l, \beta, q}^f z_{l, \beta, q}^f) + \sum_{l, \beta, q} \lambda_{l, \beta, q} z_{l, \beta, q}$. The sum $\sum_{\alpha, \beta}$ is over all α, β such that $D_{\alpha, \beta} \in D^{(\xi)} \cup D_{in}^{(\xi)}$, $D_{\alpha, \beta}^{\varphi} \in D_{\varphi}^{(\xi)} \cup D_{\varphi, in}^{(\xi)}$, and the sum $\sum_{l, \beta, q}$ is over all l, β, q such that $D_{l, \beta, q} \in D_s^{(\xi)}$, $\Delta_{l, \beta, q}^F \in D_F$, $\Delta_{l, \beta, q}^G \in D_G$, $\Delta_{l, \beta, q}^f \in D_f$; $\lambda_{\alpha, \beta}^{\varphi}$, $\lambda_{\alpha, \beta}$, $\lambda_{l, \beta, q}^F$, $\lambda_{l, \beta, q}^G$, $\lambda_{l, \beta, q}^f$, $\lambda_{l, \beta, q}$ are weights. The sets $D_{\varphi}^{(\xi)}$, $D^{(\xi)}$, and $D_{\varphi, in}^{(\xi)}$, $D_{in}^{(\xi)}$ consist of matrices $D_{\alpha, \beta}^{\varphi}$, $D_{\alpha, \beta}$, the sets $D_s^{(\xi)}$ consist of matrices $D_{l, \beta, q}$ from conditions (5), (7), and (9) whom the vector $\tilde{\tau}^{(\xi)}$, $\xi > 1$, is to satisfy.

Coordinates of the weights correspond to columns of the matrices $D_{\alpha, \beta}^{\varphi}$, $D_{\alpha, \beta}$, $\Delta_{l, \beta, q}^F$, $\Delta_{l, \beta, q}^G$, $D_{l, \beta, q}$ and vectors $\Delta_{l, \beta, q}^f$. Suppose a column of a matrix or a vector is found γ times; then the larger γ the greater role of this column or vector in minimization of the number of communications between processors and in improvement of locality. Thus, the value of appropriate coordinate is to be larger. The weights can also express the preference for the choice of operation and data allocation. Suppose it is desirable that there is no exchange of elements of some array a_{l_0} , then the weights $z_{l_0, \beta, q}^F$, $z_{l_0, \beta, q}^G$, $z_{l_0, \beta, q}^f$ are to be larger then the others.

To find the vectors $\tilde{\tau}^{(\xi)}$ it is necessary to minimize values of the variables $z_{\alpha, \beta}^{\varphi}$, $z_{\alpha, \beta}$, $z_{l, \beta, q}^F$, $z_{l, \beta, q}^G$, $z_{l, \beta, q}^f$, $z_{l, \beta, q}$. Thus, to find these vectors is to solve the following optimization problem. Choose a vector $s_{\beta}^{(\xi)} \in S_{\beta}^{(\xi)}$, $\beta \in L^{(\xi)}$, and minimize the value of the function ρ , the following condition being valid: condition (3) for $\beta \in L^{(\xi)}$, conditions (6) if $\xi \leq r$, and conditions (5), (7), (9).

The following procedure summarizes the previous investigations. The aim of the procedure is to find a multi-dimensional scheduling and data allocation satisfying the condition of communication-free allocation and the condition of space and time localization. The procedure is recursive and consists of n recursions. The ξ th recursion results in getting a vector $\tilde{\tau}^{(\xi)}$.

Procedure (finding scheduling and allocation functions): Put $\xi = 1$.

Step 1. Choose a vector $s_\beta^{(\xi)} \in S_\beta^{(\xi)}$, $\beta \in L^{(\xi)}$. Find a vector $\tilde{\tau}^{(\xi)}$ by solving the optimization problem

$$\min \left\{ \rho(z_{\alpha,\beta}^\varphi, z_{\alpha,\beta}, z_{l,\beta,q}^F, z_{l,\beta,q}^G, z_{l,\beta,q}^f, z_{l,\beta,q}) \mid \begin{array}{l} \text{condition (3), } \beta \in L^{(\xi)}, \\ \text{condition (5), } D_{\alpha,\beta}^\varphi \in D_\varphi^{(\xi)}, D_{\alpha,\beta} \in D^{(\xi)}, \\ \text{condition (7), } D_{\alpha,\beta}^\varphi \in D_{\varphi, in}^{(\xi)}, D_{\alpha,\beta} \in D_{in}^{(\xi)}, \\ \text{condition (6), } \Delta_{l,\beta,q}^G \in D_G, \Delta_{l,\beta,q}^f \in D_f, \Delta_{l,\beta,q}^F \in D_F, \xi \leq r, \\ \text{condition (9), } D_{l,\beta,q} \in D_s^{(\xi)} \end{array} \right\}.$$

Step 2. If $\xi < r + 1$ then define sets: $D_\varphi^{(\xi+1)} = D_\varphi^{(\xi)}$, $D^{(\xi+1)} = D^{(\xi)}$.

If $\xi \geq r + 1$ then define sets:

$$D_\varphi^{(\xi+1)} = D_\varphi^{(\xi)} \setminus \{ D_{\alpha,\beta}^\varphi \mid z_{\alpha,\beta}^\varphi > 0 \}, \quad D^{(\xi+1)} = D^{(\xi)} \setminus \{ D_{\alpha,\beta} \mid z_{\alpha,\beta}^\varphi > 0 \}.$$

Step 3. Define sets:

$$\begin{aligned} D_{\varphi, in}^{(\xi+1)} &= D_{\varphi, in}^{(\xi)} \setminus \{ D_{\alpha,\beta}^\varphi \mid z_{\alpha,\beta}^\varphi > 0 \}, \quad D_{in}^{(\xi+1)} = D_{in}^{(\xi)} \setminus \{ D_{\alpha,\beta} \mid z_{\alpha,\beta}^\varphi > 0 \}, \\ D_s^{(\xi+1)} &= \{ D_{l,\beta,q} \in D_s^{(1)} \mid \text{rang } T_{l,\beta,q}^{(\xi)} < r(l, \beta, q) \}. \end{aligned}$$

Step 4. Define a set $L^{(\xi+1)} = \{ \beta \mid n - \xi = n_\beta - \text{rang } T_{1;\xi}^{(\beta)} \}$.

Step 5. If $\xi = n$ then go out the procedure else increase ξ by 1 and go to step 1.

6 Data Exchange Sequence

Suppose for some fixed parameters l, β, q, ξ the conditions of communication-free allocation are not valid (i.e., even one of variables $z_{l,\beta,q}^F, z_{l,\beta,q}^G, z_{l,\beta,q}^f$ is not equal to zero at some recursion of the procedure). Then it is necessary to pass elements of array a_l for using them for the q -th input of elements of array a_l into instruction S_β .

By $P(z_1, \dots, z_r)$ denote a processor allocated at the point (z_1, \dots, z_r) of virtual processors space. According to the functions $\bar{d}^{(l)}$ and $\bar{t}^{(\beta)}$, the array elements $a_l(\bar{F}_{l,\beta,q}(J))$ are stored in the local memory of the processors $P(d_1^{(l)}(\bar{F}_{l,\beta,q}(J)), \dots, d_r^{(l)}(\bar{F}_{l,\beta,q}(J)))$ and they are used in the processors $P(t_1^{(\beta)}(J), \dots, t_r^{(\beta)}(J))$ at the iterations $(t_{r+1}^{(\beta)}(J), \dots, t_n^{(\beta)}(J))$. In the general case, point-to-point communications can be organized between pairs of these processors.)

For the program execution time to be smaller it is desirable to determine prompt communications such as broadcast, gather, scatter, reduction, and data translation. Consider for example broadcast.

Let F be an element of the set W_l . Denote by $V_{l,\beta,q}^{(F)} = \{ J \in V_\beta \mid \bar{F}_{l,\beta,q}(J) = F \}$ the set of such iterations of the initial loop nest that the array element $a_l(F)$ is used at them for the q -th input of elements of array a_l into instruction S_β .

The set $V_{l,\beta,q}^{(F)}$ is called non-degenerate if $\dim(\ker F_{l,\beta,q}) \neq 0$ and there exists a vector $J_0 \in V_{l,\beta,q}^{(F)}$ such that $J_0 + u_i \in V_\beta$, where u_i is any base vector of the intersection $\ker F_{l,\beta,q}$ and \mathbb{Z}^{n_β} .

Let $u_{l,\beta,q}^{(1)}, \dots, u_{l,\beta,q}^{(\zeta(l,\beta,q))}$ be a fundamental system of solutions of a uniform system of equations $F_{l,\beta,q}x = 0$.

Theorem 2. Suppose the set $V_{l,\beta,q}^{(F)}$ is non-degenerate; the function $\bar{F}_{l,\beta,q}$ occurs in the right part of the instruction S_β ; conditions

$$\tau^{(\beta,\xi)} u_{l,\beta,q}^{(\zeta)} = 0, \quad r+1 \leq \xi \leq n, \quad 1 \leq \zeta \leq \zeta(l,\beta,q),$$

and one of the following conditions are valid:

- a) the elements of array a_l occur only in the right parts of the instructions,
- b) constraints

$$\Phi_{\alpha,\beta} u_{l,\beta,q}^{(\zeta)} = 0, \quad 1 \leq \zeta \leq \zeta(l,\beta,q),$$

are valid for the flow-dependence produced by the q -th input of elements of array a_l into instruction S_β .

Then to pass the data $a_l(F)$ it is possible to arrange broadcast from the processor $P(d_1^{(l)}(\bar{F}_{l,\beta,q}(J)), \dots, d_r^{(l)}(\bar{F}_{l,\beta,q}(J)))$ to the processors $P(t_1^{(\beta)}(J), \dots, t_r^{(\beta)}(J))$ at the iteration $(t_{r+1}^{(\beta)}(J), \dots, t_n^{(\beta)}(J))$, $J \in V_{l,\beta,q}^{(F)}$.

7 Conclusion

In this paper, we propose a method of mapping algorithms for parallel execution onto distributed memory parallel computers. The method provides with determination of operation and data allocation over processors, an execution sequence of operations, and data exchange necessary for the program execution. The aim is to minimize a number of communications, to improve locality of an algorithm, and to determine the possibility of broadcasts.

Note some advantages of the method suggested:

- an initial algorithm is represented by affine loop nests of an arbitrary nesting structure;
- the suggested conditions can be simply obtained from a source algorithm;
- the conditions do not depend on the definite values of outer variables; the obtained functions depend on outer variables parametrically;
- the method can be automated.

The method was applied for mapping algorithms for matrix transformations onto distributed memory parallel computers. These algorithms was implemented on the supercomputer SKIF (it is located at NAS of Belarus, Minsk).

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